J. Fluid Mech. (1973), *vol.* 59, *part* 1, *pp.* 117–128
Printed in Great Britain

On optimum profiles in Stokes flow

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(Received 1 December 1972)

In this paper, we obtain the first-order necessary optimality conditions of an optimal control problem for a distributed parameter system with geometric control, namely, the minimum-drag problem in Stokes flow (flow at a very low Reynolds number). We find that the unit-volume body with smallest drag must be such that the magnitude of the normal derivative of the velocity of the fluid is constant on the boundary of the body. In a three-dimensional uniform flow, this condition implies that the body with minimum drag has the shape of a pointed body similar in general shape to a prolate spheroid but with some differences including conical front and rear ends of angle 120°.

1. Introduction

Despite the simplicity of the partial differential equations that describe the motion of a viscous fluid at zero Reynolds number, it has not been possible, so far, to discover the shape of the body of given volume which produces minimum drag when moving slowly through a viscous fluid at constant speed. Among other attempts at solving the problem, we recall the works of Watson (1971) and Tuck (1968). By merging the variational maximum principle for the Stokes equations and a numerical minimization procedure, Watson obtained an algorithm which selects the best shape out of a given family of surfaces depending on a finite number of parameters. Unfortunately, Watson did not try it on surfaces with sharp ends. By assuming that the stream function for an arbitrary axisymmetric body can be expressed in terms of point sources and 'Stokeslets', Tuck derived a pair of nonlinear integral equations which characterize the optimum shape. However, owing to their complexity, these equations remain unsolved.

In the theory of calculus of variations, the minimum-drag problem in Stokes flow is classified as an 'optimal control problem for a distributed parameter system, the control being a geometric element of the system'. Lions (1972) obtained an existence theorem for a problem of similar nature, but, as far as we know, first-order necessary optimality conditions for this problem have not been given. Hence, part of the interest of this paper lies in its contribution to optimal control theory, while part is of direct fluid-mechanical interest. We preferred to present the results for both fields in one paper only, essentially because the generalization of the results to more complex optimal control problems is straightforward.

The subject treated in this paper is at the border of two research fields; optimum problems in Stokes flow are indeed mathematically difficult and because of this we could not avoid use of the theory of partial differential equations in weak form and of their Sobolev spaces. Hence, after having stated the problem in § 2, we begin by recalling some of the properties of the weak solutions of the Stokes equations (see § 3). In § 4, we obtain the first-order necessary optimality conditions of the above optimal shape of the Stokes minimum-drag problem under volume constraints. These conditions are simple and lead naturally to the construction of numerical methods for solving them. This is done in § 5 but, lacking a numerical subroutine to integrate the Stokes equations in an unbounded domain, we have not programmed the algorithm derived. However, § 5 contains an argument which enables us to conjecture (to a good approximation) the optimal shape. Lastly, in § 6, we examine a few other minimum-drag problems in Stokes flow. In particular, we give the optimality conditions for the body of unit surface area which has minimum drag.

2. Statement of the problem

Consider the optimal control problem

$$\min_{S\in\mathcal{S}}\left\{\int_\Omega \frac{1}{2}\sum_{i,j=1}^{3}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)^2 d\Omega \,\middle|\, \nabla^2\mathbf{U} = \nabla p, \nabla . \mathbf{U} = 0 \text{ almost everywhere in } \Omega; \; \mathbf{U}|_S = 0, \mathbf{U}|_\Gamma = \mathbf{z}\right\}, \quad (2.1)$$

where $\mathcal{S}$ is a given subset of the set of almost everywhere infinitely continuously differentiable surfaces in R^3; Ω is the open set of R^3 with boundary $\partial\Omega = \Gamma \cup S$; $\mathbf{U} = (u_1, u_2, u_3)$ is a weak solution (see next section) of the partial differential equations above (the Stokes equations with viscosity one); and $\mathbf{z}$ is a given function in $H^{\frac{1}{2}}(\Gamma)$.†

In particular, if $\mathcal{S}$ is the set of boundaries of bodies with unit volume, and Γ and $\mathbf{z}$ are as in figure 1, for large Ω (and S centrally located), problem (2.1) approaches the Stokes minimum-drag problem. Indeed, the partial differential equations in (2.1) describe the motion of a viscous fluid with speed $\mathbf{U}(\mathbf{x})$ and pressure $p(\mathbf{x})$ at $\mathbf{x}\in\Omega$, in the (stationary) Stokes approximation (low Reynodsl number); and the cost in (2.1) is the rate of energy $\mathscr{E}$ dissipated by the fluid, which in our case is related to the magnitude of the drag force F on S, by the formula

$$\mathscr{E} = U_0 F + \text{higher order terms in } d + \text{terms independent of } S, \quad (2.2)$$

where U_0 is the magnitude of the (uniform) speed in the fluid far from S, d is the ratio of the volume enclosed by S to that enclosed by Ω, when S is 'centrally' located in Ω.

Since (2.1) is an optimal control problem almost in the form studied by Lions (1968), we shall use some of the standard techniques of the field. Therefore we shall begin by recalling some properties of the partial differential equations in weak form.

† $H^m(\Gamma)$ is the Sobolev space of order m on Γ (see Lions 1968, p. 39). It suffices here to know that the speed distribution $\mathbf{z}$ in figure 1 belongs to $H^{\frac{1}{2}}(\Gamma)$.

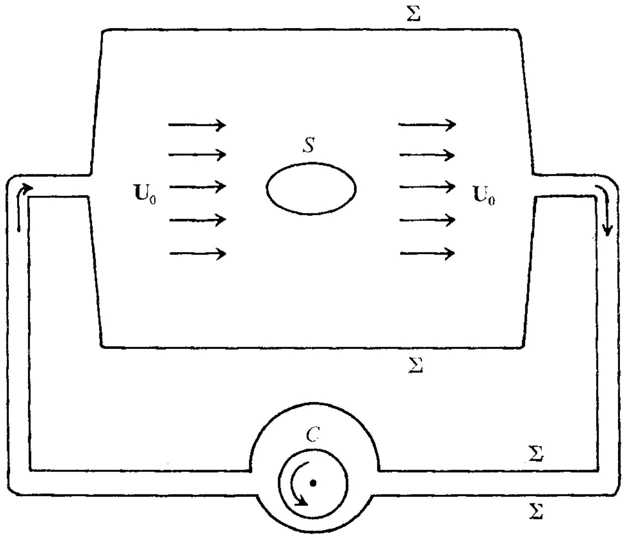

FIGURE 1. A possible design for the study of the drag on S. The fluid inside Σ is maintained at slow motion by the pump C (a rotating cylinder for example). S is very small compared with Σ so that the motion is almost uniform far from S. In this case, $\Gamma = \Sigma \cup C$, $\mathbf{z} = 0$ on Σ, $\mathbf{z} = \boldsymbol{\omega} \times x$, at $\mathbf{x} \in C$, where $\boldsymbol{\omega}$ is the angular velocity of C.

3. The Stokes equations in weak form

Let $C_n^m(\Omega)$ be the space of m times continuously differentiable functions from Ω into R^n. Suppose that $\mathbf{U} \in C_3^1(\Omega)$ and $p \in C_1^1(\Omega)$ are such that

$$\nabla^2 \mathbf{U} = \nabla p, \quad \nabla . \mathbf{U} = 0 \quad \text{everywhere in } \Omega. \tag{3.1}$$

Then, by multiplying (3.1) by $\boldsymbol{\phi} \in D_3^1(\Omega)$, where

$$D_3^1(\Omega) = \{\boldsymbol{\phi} \in C_3^\infty(\Omega) | \overline{\{\mathbf{x} | \boldsymbol{\phi}(\mathbf{x}) \neq 0\}} \subset \Omega, \nabla . \boldsymbol{\phi} = 0\}, \tag{3.2}$$

and by integrating by parts, one obtains

$$\int_\Omega \mathbf{U} . \nabla^2 \boldsymbol{\phi} \, d\Omega = 0. \tag{3.3}$$

Conversely, if $\mathbf{U}$ satisfies (3.3) for all $\boldsymbol{\phi} \in D_3^1(\Omega)$ and

$$\int_{\Gamma \cup S} \mathbf{U} . \boldsymbol{\psi} \, d\Gamma = \int_\Gamma \mathbf{z} . \boldsymbol{\psi} \, d\Gamma \quad \text{for all} \quad \boldsymbol{\psi} \in C_3^\infty(\Gamma \cup S) \quad \text{with} \quad \int_{\Gamma \cup S} \boldsymbol{\psi} . \mathbf{dS} = 0, \tag{3.4}$$

then $\mathbf{U}$ satisfies (3.1) almost everywhere $\nabla^2 \mathbf{U}$ exists, and is said to be the weak solution of

$$\nabla^2 \mathbf{U} = \nabla p, \quad \nabla . \mathbf{U} = 0 \quad \text{in} \quad \Omega; \quad \mathbf{U}|_S = 0, \quad \mathbf{U}|_\Gamma = \mathbf{z}. \tag{3.5}$$

Let $\tilde{\mathbf{z}}$ be an extension in $H_3^2(\Omega)$† of $\mathbf{z}$ such that $\nabla.\tilde{\mathbf{z}} = 0$ in Ω (there exist such extensions; see Ladyzhenskaya 1963) and let $\mathbf{V}$ be a solution of the variational equation

$$\int_\Omega \sum_{i,j=1}^{3} \frac{\partial v_i}{\partial x_j}\frac{\partial \phi_i}{\partial x_j}\, d\Omega = \int_\Omega \boldsymbol{\phi}.\nabla^2\tilde{\mathbf{z}}\, d\Omega \quad \text{for all} \quad \boldsymbol{\phi}\in\{\boldsymbol{\phi}\in H_3^2(\Omega)\,|\nabla.\boldsymbol{\phi} = 0;\ \boldsymbol{\phi}|_{\partial\Omega} = 0\}. \tag{3.6}$$

It is easy to show that (3.6) has a unique solution $\mathbf{V}$ in $H_3^2(\Omega)$ (see theorem I.1.2 in Lions 1968) with $V|_{\partial\Omega} = 0$ and $\nabla.\mathbf{V} = 0$. Hence $\mathbf{V}+\tilde{\mathbf{z}}$ is the unique weak solution, in $H_3^2(\Omega)$, of (3.5).

4. Optimality conditions for problem (2.1)

Problem (2.1) is a problem of optimal control of a system governed by a linear elliptic partial differential equation and with quadratic cost, but the control is a geometric element of the system. Therefore, we must face two difficulties.

(i) The control space is not (*a priori*) a linear space.

(ii) The solution $\mathbf{U}^S$ to the partial differential equation is not a 'linear' function of the control S.

Instead of trying to solve (2.1) directly, by giving a linear structure to $\mathscr{S}$ (Hausdorff metric for example), we shall relate it to the problem

$$\min_{\mathbf{w}\in W}\left\{\int_\Omega \frac{1}{2}\sum_{i,j=1}^{3}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)^2 d\Omega\,|\nabla^2\mathbf{U} = \nabla p;\quad \nabla.\mathbf{U} = 0;\quad \mathbf{U}|_S = \mathbf{w},\quad \mathbf{U}|_\Gamma = \mathbf{z}\right\}, \tag{4.1}$$

for which a solution is known (see Lions 1968, chap. 2). The remaining difficulty, i.e. the effect of a small change of body shape from S into S' on the distribution $\mathbf{w}$ of fluid speed on S, will be solved by means of a Taylor expansion, made possible from the assumption $\mathbf{z}\in H^{\frac{5}{2}}(\Gamma)$.

Thus we shall prove the following theorem.

THEOREM 1. Suppose that, in (2.1), $\mathbf{z}\in H^{\frac{5}{2}}(\Gamma)$ and S is parametrized by $\mathbf{s}\in[0, 1]^2$. Then

$$S = \{\mathbf{x}(\mathbf{s})|\mathbf{s}\in[0, 1]^2\}\in\mathscr{S} \tag{4.2}$$

is a solution of (2.1) only if

$$\int_S \left\|\frac{\partial \mathbf{U}^S}{\partial n}\right\|^2 \alpha(\mathbf{s})\, d\mathbf{s} + \sigma(\alpha) \geqslant 0 \quad \text{for all admissible } \alpha\text{'s}, \tag{4.3}$$

where $\partial\mathbf{U}^S/\partial n$ is the derivative of the speed distribution $\mathbf{U}^S$, the weak solution of (3.5), along the outward normal $\mathbf{n}$ to S; the set of admissible α's is

$$\{\alpha|\{\mathbf{x}(\mathbf{s})+\mathbf{n}(\mathbf{s})\,\alpha(\mathbf{s})|\ \mathbf{s}\in[0, 1]^2\}\in\mathscr{S}\}$$

and $\sigma(\alpha)$ is such that $\lim_{\alpha\to 0}\{|\sigma(\alpha)|/\|\alpha\|_{C^{\frac{3}{2}}[0,1]^2}\} = 0$.

† $H_3^l(\Omega) = \left\{\boldsymbol{\phi}\in L_3^2(\Omega)\,|D^{\mathbf{p}}\phi\in L_3^2(\Omega) \text{ for all } \mathbf{p} = (p_1, \ldots, p_n) \text{ with } \sum_{l=1}^{n} p_l = l\right\}$, where the derivatives $D^{\mathbf{p}} = \partial^{p_1+p_2\cdots p_n}/\partial x_1^{p_1}\ldots\partial x_n^{p_n}$ are in the distributions sense.

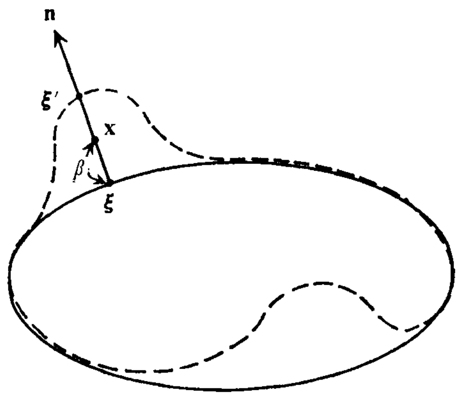

FIGURE 2. If S (continuous line) is parameterized by $\mathbf{s} \in [0, 1]^2$, a given function α: $[0, 1]^2 \to R$ defines a perturbation S' (dotted line) by the formula
$$S' = \{\boldsymbol{\xi}'(\mathbf{s}) | \boldsymbol{\xi}'(\mathbf{s}) = \boldsymbol{\xi}(\mathbf{s}) + \alpha(\mathbf{s})\, \mathbf{h}(\mathbf{s}), \mathbf{s} \in [0, 1]^2\}.$$

Proof. In order to simplify our equations, we introduce the notation

$$E^S(\mathbf{x}) = \frac{1}{2} \sum_{i,j=1}^{3} \left(\frac{\partial u_i^S(\mathbf{x})}{\partial x_j} + \frac{\partial u_j^S(\mathbf{x})}{\partial x_i} \right)^2, \quad \mathbf{x} \in \Omega_S, \tag{4.4}$$

where $\mathbf{U}^S = (u_1^S, u_2^S, u_3^S)$ is the weak solution of (3.5).

Clearly, S is a solution of the optimization problem (2.1) only if

$$\int_{\Omega_{S'}} E^{S'}(\mathbf{x})\, d\Omega \geqslant \int_{\Omega_S} E^S(\mathbf{x})\, d\Omega \quad \text{for all} \quad S' \in \mathscr{S}. \tag{4.5}$$

$\mathbf{U}^{S'}$ is defined only outside S' but it is shown in the appendix that we can extend the definition of $\mathbf{U}^{S'}(\mathbf{x})$ to all points $\mathbf{x}$ inside S', but outside S, and keep the property $\mathbf{U} \in H_3^3(\Omega_S \cup \Omega_{S'})$. Hence (4.5) can be rewritten as

$$\int_{\Omega_S} (E^{S'}(\mathbf{x}) - E^S(\mathbf{x}))\, d\Omega - \left[\int_{\Omega_S - \Omega_{S'} \cap \Omega_S} E^{S'}(\mathbf{x})\, d\Omega - \int_{\Omega_{S'} - \Omega_{S'} \cap \Omega_S} E^{S'}(\mathbf{x})\, d\Omega \right] \geqslant 0$$
$$\text{for all} \quad S' \in \mathscr{S}. \tag{4.6}$$

Let $\mathbf{n}$ be the outward normal to S at $\boldsymbol{\xi}$. Let $\boldsymbol{\xi}'$ be the intersection of $\mathbf{n}$ and S'; let α and β be such that (see figure 2)

$$\mathbf{x} = \boldsymbol{\xi} + \mathbf{n}\beta, \quad \boldsymbol{\xi}' = \boldsymbol{\xi} + \mathbf{n}\alpha. \tag{4.7}$$

From (4.7), there is a one-to-one correspondence between the surface S' (close to S) and the functions α: $[0, 1]^2 \to R$. We shall now relate the first and second terms in (4.6) with $\alpha(\ )$.

According to corollary I.9.1 in Lions & Magenes (1967), every function in $H_3^3(\Omega)$ is almost everywhere continuously differentiable.† Therefore, from the mean-value theorem for integrals,

$$\int_{\Omega_S - \Omega_{S'} \cap \Omega_S} E^{S'}(\mathbf{x})\, d\Omega - \int_{\Omega_{S'} - \Omega_{S'} \cap \Omega_S} E^{S'}(\mathbf{x})\, d\Omega = \int_S E^{S'}(\boldsymbol{\xi}(\mathbf{s}))\, \alpha(\mathbf{s})\, dS + \sigma(\alpha, \dot{\alpha}, \ddot{\alpha}). \tag{4.8}$$

† This is why we require $\mathbf{z} \in H^{\frac{5}{2}}(\Gamma)$.

The first term of (4.6) can be evaluated as follows. Let $W = \{\mathbf{U}^{S'}|_S \,|S' \in \mathscr{S}\}$ and let $\mathscr{E}\colon W \to R$ be defined by

$$\mathscr{E}(\mathbf{w}) = \int_{\Omega_S} \frac{1}{2}\sum_{i,j=1}^{3} \left(\frac{\partial u_i^w}{\partial x_j} + \frac{\partial u_j^w}{\partial x_i}\right)^2 d\Omega, \tag{4.9}$$

where $(u_1^w, u_2^w, u_3^w) = \mathbf{U}^w$ is a weak solution of

$$\nabla^2\mathbf{U} = \nabla p, \quad \nabla.\mathbf{U} = 0 \quad \text{in} \quad \Omega; \quad \mathbf{U}|_S = \mathbf{w} \in W, \quad U|_\Gamma = \mathbf{z}. \tag{4.10}$$

Then

$$\int_{\Omega_S} [E^{S'}(\mathbf{x}) - E^S(\mathbf{x})]\, d\Omega = \mathscr{E}(\mathbf{w}') - \mathscr{E}(0), \tag{4.11}$$

where $\mathbf{w}' = \mathbf{U}^{S'}|_S$. $\mathscr{E}(\ )$ is a quadratic continuous function from $H^{\frac{1}{2}}(S)$; therefore, its variations can be evaluated as follows (see Lions 1968, chap. 2):

$$\mathscr{E}(\mathbf{w}') - \mathscr{E}(\mathbf{w}) = \int_{\Omega} \sum_{i,j=1}^{3} \left(\frac{\partial}{\partial x_i}(u_j^{w'} - u_j^w) + \frac{\partial}{\partial x_j}(u_i^{w'} - u_i^w)\right)\left(\frac{\partial u_i^w}{\partial x_j} + \frac{\partial u_j^w}{\partial x_i}\right) d\Omega + \sigma(\mathbf{w}' - \mathbf{w}). \tag{4.12}$$

By integrating the right-hand side by parts, (4.12) becomes

$$-\int_{\Omega} 2 \sum_{i,j=1}^{3} \left(\frac{\partial^2 u^w}{\partial x_j^2} + \frac{\partial^2 u^w}{\partial x_i\, \partial x_j}\right)(u_i^{w'} - u_i^w)\, d\Omega$$
$$+ 2\int_{\partial\Omega} \sum_{i,j=1}^{3} \left(\frac{\partial u_i^w}{\partial x_j} + \frac{\partial u_j^w}{\partial x_i}\right)(u_j^{w'} - u_j^w)\, d(\partial\Omega)_i + \sigma(\mathbf{w}' - \mathbf{w}). \tag{4.13}$$

As $\nabla.\mathbf{U} = 0$ we find that $\sum_{j=1}^{3} \dfrac{\partial^2 u_j}{\partial x_i\, \partial x_j} = 0$; hence (4.13) becomes

$$-\int_{\Omega} 2\nabla^2\mathbf{U}^w.(\mathbf{U}^{w'} - \mathbf{U}^w)\, d\Omega - \int_{S\cup\Gamma} 2\boldsymbol{\sigma}^w(\mathbf{U}^{w'} - \mathbf{U}^w)\, d\mathbf{S} + \sigma(\mathbf{w}' - \mathbf{w}), \tag{4.14}$$

where $\sigma_{ij} = \partial u_i/\partial x_j + \partial u_j/\partial x_i$.

By making use of $\nabla^2\mathbf{U} = \nabla p$ and by integrating by parts, (4.14) becomes

$$\int_{\Omega} 2p^w\nabla.(\mathbf{U}^{w'} - \mathbf{U}^w)\, d\Omega - 2\int_{S\cup\Gamma} (-p^w\mathbf{I} + \boldsymbol{\sigma}^w)(\mathbf{U}^{w'} - \mathbf{U}^w)\, d\mathbf{S} + \sigma(\mathbf{w}' - \mathbf{w}). \tag{4.15}$$

Hence, since $\nabla.\mathbf{U} = 0$ and $\mathbf{U}^{w'} = \mathbf{U}^w$ on Γ and $\mathbf{U}^{w'} - \mathbf{U}^w = \mathbf{w}' - \mathbf{w}$ on S,

$$\mathscr{E}(\mathbf{w}') - \mathscr{E}(\mathbf{w}) = -2\int_S (-p^w\mathbf{I} + \boldsymbol{\sigma}^w)(\mathbf{w}' - \mathbf{w})\, d\mathbf{S} + \sigma(\mathbf{w}' - \mathbf{w}). \tag{4.16}$$

Now, $\mathbf{U}^{S'}$ belongs to $H_3^3(\Omega)$, hence it also belongs to $C_3^1(\overline{\Omega})$ (corollary 9.1 in Lions & Magenes 1967). Therefore from a Taylor expansion

$$\mathbf{w}' = \mathbf{U}^{S'}|_S = \mathbf{U}^{S'}|_{S'} - \frac{\partial \mathbf{U}^{S'}}{\partial n}\bigg|_S \alpha + \sigma(\alpha) = -\frac{\partial \mathbf{U}^{S'}}{\partial n}\bigg|_S \alpha + \sigma(\alpha). \tag{4.17}$$

Hence from (4.11), (4.16) and (4.17)

$$\int_{\Omega_S} E^{S'}(\mathbf{x})\, d\Omega - \int_{\Omega_S} E^S(\mathbf{x})\, d\Omega = +2\int_S (-p^S\mathbf{I} + \boldsymbol{\sigma}^S)\frac{\partial \mathbf{U}^{S'}}{\partial n}\alpha\, d\mathbf{S} + \sigma(\alpha). \tag{4.18}$$

It is shown in the appendix that $[\partial \mathbf{U}^{S'}/\partial n]_S$ is weakly continuous in α, therefore (4.6), (4.8) and (4.18) imply that

$$2\int_S (-p^S \mathbf{I} + \boldsymbol{\sigma}^S) \frac{\partial \mathbf{U}^S}{\partial n} \alpha \, d\mathbf{S} + \sigma(\alpha) \geqslant \int_S \alpha E^S(\mathbf{x}) \, dS + \sigma'(\alpha) \tag{4.19}$$

for all admissible α.

Now, $\mathbf{U}^S|_S = 0$ and $\nabla . \mathbf{U}^S = 0$ imply that

$$E^S(\mathbf{x}) = \|\partial \mathbf{U}^S/\partial n\|^2 \quad \text{at} \quad \mathbf{x} \in S, \tag{4.20}$$

$$\frac{\partial \mathbf{U}}{\partial n} d\mathbf{S} = \frac{\partial \mathbf{U}_n}{\partial n} d\mathbf{S} = 0 \quad \text{on} \quad S, \tag{4.21}$$

$$\boldsymbol{\sigma} \frac{\partial \mathbf{U}}{\partial n} d\mathbf{S} = \left\| \frac{\partial \mathbf{U}}{\partial n} \right\|^2 dS \quad \text{on} \quad S. \tag{4.22}$$

Hence (4.20) becomes

$$\int_S \left\| \frac{\partial \mathbf{U}^S}{\partial n} \right\|^2 \alpha \, dS + \sigma(\alpha) \geqslant 0 \quad \text{for all admissible } \alpha, \tag{4.23}$$

which completes our proof.

COROLLARY 1. *If $\mathscr{S}$ is the set of bodies of unit volume, S is optimal for* (2.1) *only if $\|\partial \mathbf{U}/\partial n\|^S$ is constant almost everywhere on S.*

Proof. For ϵ, $\mathbf{s}'$ and $\mathbf{s}''$ given, define $\alpha_m(\ )$ by

$$\alpha_m(\mathbf{s}) = \begin{cases} \epsilon m^{-1} \exp[(m^{-2} - \rho^2)^{-1}] \text{ on } \mathbf{s} = \mathbf{s}' + \boldsymbol{\rho}(\cos\theta, \sin\theta), \\ \qquad\qquad |\boldsymbol{\rho}| \leqslant m^{-1}, \quad \theta \in [0, 2\pi], \\ -\epsilon m^{-1} \exp[(m^{-2} - \rho^2)^{-1}] \text{ on } \mathbf{s} = \mathbf{s}'' + \boldsymbol{\rho}(\cos\theta, \sin\theta), \\ \qquad\qquad |\boldsymbol{\rho}| \leqslant m^{-1}, \quad \theta \in [0, 2\pi], \\ 0 \quad \text{otherwise.} \end{cases} \tag{4.24}$$

Up to second-order terms in ϵ/m, α is admissible, in particular it allows almost no change of volume; hence, when $m \to \infty$ equation (4.23) becomes

$$\epsilon \left[\left\| \frac{\partial \mathbf{U}}{\partial n}(\mathbf{s}') \right\|^2 - \left\| \frac{\partial \mathbf{U}}{\partial n}(\mathbf{s}'') \right\|^2 \right] \geqslant 0 \tag{4.25}$$

for all ϵ, $\mathbf{s}'$ and $\mathbf{s}''$ for which $\partial \mathbf{U}/\partial n$ is continuous. Upon changing ϵ into $-\epsilon$ equation (4.25) becomes

$$\left\| \frac{\partial \mathbf{U}}{\partial n}(\mathbf{s}') \right\| = \left\| \frac{\partial \mathbf{U}}{\partial n}(\mathbf{s}'') \right\| \quad \text{for almost all } \mathbf{s}', \mathbf{s}''. \tag{4.26}$$

5. Approach to the unit-volume body with minimum drag in Stokes flow

It is beyond the scope of this paper to discuss whether the previous computations remain valid for unbounded domains Ω; we shall assume this, although it is not essential for the following discussion. Thus the body S with unit volume and smallest drag in a uniform fluid in slow motion $\mathbf{U}_0$ must satisfy

$$\|\partial \mathbf{U}/\partial n\| = \text{constant almost everywhere on } S, \tag{5.1}$$

where $\mathbf{U}$ is the weak solution of

$$\nabla^2\mathbf{U} = \nabla p, \quad \nabla.\mathbf{U} = 0; \quad \mathbf{U}|_S = 0, \quad \mathbf{U}_\infty = \mathbf{U}_0. \tag{5.2}$$

The problem of finding the S's that satisfy (5.1) and (5.2) and enclose unit volume is far from being trivial. We made several attempts at obtaining a solution of (5.1) and (5.2) in closed form: all of them failed. However, before proceeding any further, we shall make the following comments: if (5.1) and (5.2) has a unique solution S, then (i) S is axisymmetric and has a centre of symmetry (since (5.1) and (5.2) are invariant under rotations and a change of sign of $\mathbf{U}_0$); (ii) the front and rear ends of S must each be tangential to a cone of angle 120°.† Indeed, if they are smooth, $\partial\mathbf{U}/\partial n = 0$ at those points; if they are shaped like cusps $\|\partial\mathbf{U}/\partial n\| = +\infty$. If the front end of S is a cone of angle θ_1, let ψ be the stream function of the problem‡ and let the origin be at the front end of S, then $\psi|_S = 0$ implies that the first-order term of the Taylor expansion, in r, of ψ has the form $r^n f(\theta)$, and we find that $cr^3(\cos\theta\sin\theta + \frac{1}{2}\sin^2\frac{1}{2}\theta)$, c constant, are the only solutions of

$$E^4(r^n f(\theta)) = 0; \quad f(\theta_1) = 0, \quad r^{n-1}f(\theta_1) = 0, \tag{5.3}$$

for which $$\|\partial\mathbf{U}/\partial n\| = c|E^2(r^n f(\theta))/r\sin\theta|$$

is constant at $\theta = \theta_1$.

Apart from those comments, any other information about S must be found with the help of numerical methods.

A quick look at the literature for similar problems in optimal control theory (see, for example, Polak (1971)) tells us that an algorithm of the following type is likely to converge to the solution of (5.1) and (5.2) if one exists.

ALGORITHM 1.

Step 0. Choose an initial body S_0 (sphere of unit volume for example); set $i = 0$.

Step 1. Compute $\mathbf{U}_i$ by solving (5.2) with $S = S_i$.

Step 2. Compute $\|\partial\mathbf{U}^{S_i}/\partial n\|^2$ on S_i.

Step 3. Set $S(\lambda) = \{\mathbf{x}|\mathbf{x} = \beta[\mathbf{x}(s) - \alpha_i(\mathbf{s})\mathbf{n}(\mathbf{s})]\}$, where $\mathbf{n}(\mathbf{s})$ is the outward normal to S_i at $\mathbf{x}(\mathbf{s})$, $\alpha_i(\mathbf{s}) = \lambda(\|\partial\mathbf{U}^{S_i}(\mathbf{x}(\mathbf{s}))/\partial n\|^2 - k_i)$, k_i is the mean value of $\|\partial\mathbf{U}^{S_i}/\partial n\|^2$ on S_i and β is such that the volume enclosed by S_i is unity.

Step 4. Compute λ_i, the solution of

$$\min_\lambda\left\{\int_{\Omega(\lambda)} E^{S(\lambda)}(\mathbf{x})\,d\Omega|\lambda\in[0,1]\right\}.§ \tag{5.4}$$

† This point is due to Sir James Lighthill.

‡ If $\psi: R^2 \to R$ is a solution of

$$E^4\psi = 0; \quad \psi|_S = 0, \quad [\partial\psi/\partial n]_S = 0, \quad \psi_\infty = \tfrac{1}{2}\varpi^2 V_0 + c';$$

where, in cylindrical co-ordinates (z, ϖ, ϕ)

$$E^2 = \varpi\frac{\partial}{\partial\varpi}\left(\frac{1}{\varpi}\frac{\partial}{\partial\varpi}\right) + \frac{\partial^2}{\partial z^2}.$$

Then, if S is axisymmetric around $\mathbf{U}_0$,

$$\mathbf{U} = \left(-\frac{1}{\varpi}\frac{\partial}{\partial\varpi}\psi, \frac{1}{\varpi}\frac{\partial\psi}{\partial z}, 0\right)$$

is a solution of (5.2).

§ This one-dimensional minimization problem can be solved by means of a golden-section search or replaced by a two-line rule (see Polak 1971, pp. 31, 36).

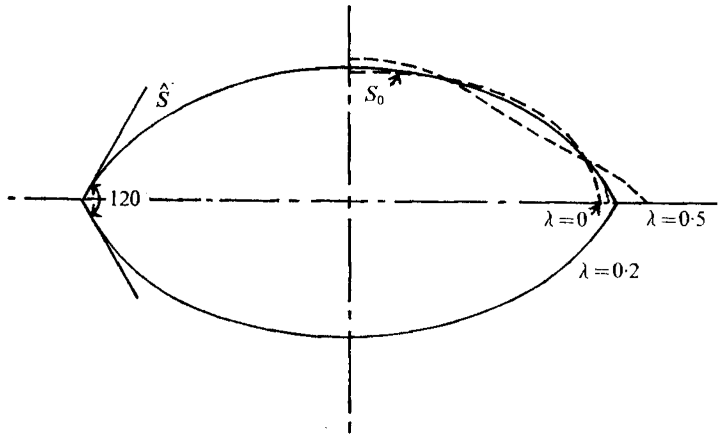

FIGURE 3. $\hat{S}$ is conjectured to lie within 5% of the solution of (5.1) and (5.2) if any exists. S_0 is the prolate spheroid with smallest drag.

Step 5. Set $S_{i+1} = S(\lambda_i)$ and go to step 1.

Algorithm 1 is a straightforward adaptation to our problem of the gradient algorithms for optimal control; it generates a sequence of bodies S_i with smaller and smaller drags. Indeed, from (4.5), (4.23) and the fact that the volume enclosed by S_{i+1} is unity

$$\int_{\Omega_{i+1}} E^{S_{i+1}}(\mathbf{x})\, d\Omega - \int_{\Omega_i} E^{S_i}(\mathbf{x})\, d\Omega = -\lambda_i \int_{S_i} \left(\left\| \frac{\partial \mathbf{U}^{S_i}}{\partial n} \right\|^2 - k_i \right) \alpha_i'\, dS + \sigma(\alpha_i, \beta - 1), \tag{5.5}$$

where $\alpha_i' = \|\partial \mathbf{U}^{S_i}/\partial n\|^2 - k_i$; this together with (5.4) and (2.2) implies that $F^{S_{i+1}} \leqslant F^{S_i}$.

Thus, it is theoretically possible to program algorithm 1 on a computer in order to find a solution of (5.1) and (5.2); however, step 3 requires an accurate knowledge of the solution of (5.2), which, in turn, is extremely difficult to achieve (although possible). Since we did not have any good subroutine to integrate (5.2), we did not make any attempt at programming algorithm 1.

However, we shall now perform by hand an iteration of algorithm 1 starting with

$$S_0 = \{\mathbf{x}(r, \theta) \mid r = (1 - 0{\cdot}74 \cos^2\theta)^{-\frac{1}{2}},\ \theta \in [0, 2\pi]\}.$$

S_0 is the prolate spheroid with smallest drag ($F = 95{\cdot}61\,\%$ of that for the sphere of equal volume) and the stream function of the flow around it is known analytically (see Happel & Brenner 1965, p. 153):

$$\|\partial \mathbf{U}/\partial n\| = 1{\cdot}04 \sin\theta (1 - 0{\cdot}74 \cos^2\theta)^{-1} \quad (\|\mathbf{U}_0\| = 1).\dagger \tag{5.6}$$

Therefore, from step 3, the body S_1 obtained from S_0 by adding onto the outward normal of S_0 the quantity in (5.7) below will be an improvement over S_0:

$$-\lambda(\|\partial \mathbf{U}/\partial n\|^2 - k_i) = \lambda[1{\cdot}15 - 1{\cdot}04 \sin^2\theta (1 - 0{\cdot}74 \cos^2\theta)^{-2}], \tag{5.7}$$

† Note that $\|\partial \mathbf{U}/\partial u\|^2$ lies within 10% of its mean on 70% of S_0. Therefore S_0 is already a good approximation to the solution of (5.1) and (5.2).

where λ is a solution of (5.4). We have drawn on figure 3 the surfaces obtained for different values of λ. The most likely value for λ, i.e. the one that gives the smoothest curve and yet fairly sharp front and rear ends, is $\lambda = 0{\cdot}2$, for which, from (5.7), the drag on the corresponding body is of the order of 91 % of the drag on the sphere of equal volume. This and the fact that gradient methods in optimal control converge generally like a geometric progression lead us to believe that the surface with $\lambda = 0{\cdot}2$, further improved by conical front and rear ends, is a good (say 5 %) approximation to the solution (if any!) of (5.1) and (5.2); its drag is probably around 90 % of that on the sphere of equal volume. Those figures are slightly above those given by Watson (1971); the shape we obtain is different from the 'flat eight' shape obtained by Watson. We must credit this difference to the fact that Watson did not try bodies with pointed ends.

6. Other minimum-drag problems

(i) The minimum-drag problem for axisymmetric unit-volume bodies at the centre of an infinitely long tube is treated in exactly the same fashion. The optimality condition is also $\|\partial \mathbf{U}/\partial n\|$ = constant on S, which implies that the solution looks like the one drawn on figure 3 but becomes more slender as the diameter of the tube becomes smaller.

(ii) Given a body Σ (not necessarily axisymmetric), can one find an outer surface S containing Σ in its interior and such that the magnitude of the drag on S is smaller than that on Σ? From theorem 1, it is straightforward to show that the velocity field for S must satisfy the Stokes equations and

$$\|\partial \mathbf{U}^S/\partial n\| = 0 \tag{6.1}$$

at almost all points of S which do not belong to Σ.

(iii) The optimality conditions for the minimum-drag problem for bodies with unit surface area can be obtained from theorem 1. One should proceed as in the proof of corollary 1 with S' obtained from S by adding a small bump on one side and replacing a part of S by a plane section somewhere else. In the case of axisymmetric bodies, the condition obtained depends on the radius of curvature R of S, i.e.

$$R\|\partial \mathbf{U}^S/\partial n\|^2 = \text{constant}$$

at almost every point of S where R is finite.

7. Conclusion

The method we have used to derive the optimality conditions above is quite natural for someone familiar with the techniques of the calculus of variations. It can be applied to problems with more complex unbounded operators, as long as it is possible to ensure that the solutions of the partial differential equation are almost everywhere continuously differentiable. However, the optimality condition will, in general, depend on the solution of the adjoint equation of the system.

In spite of the nice form of our optimality condition we have only been able to conjecture the optimal shape of the unit-volume body with smallest drag in the Stokes flow, but we hope that someone in possession of the good numerical subroutine for the Stokes equation will be interested in programming algorithm 1.

I would like to thank Sir James Lighthill and Dr D. Weihs for their extremely helpful suggestions.

Appendix

$\nabla \mathbf{U}^{S'}|_S$ is weakly continuous in α

From the definition of derivatives in the distribution sense it suffices to show that $\mathbf{U}^{S'}$ converges weakly to $\mathbf{U}^S$ when $\alpha \to 0$. Let $\mathbf{v}^{S'}$ be the solution of

$$\int_{\Omega_{S'}} \sum_{i,j=1}^{3} \frac{\partial v_i^{S'}}{\partial x_j}\frac{\partial \phi_i}{\partial x_j}\, d\Omega = \int_{\Omega_{S'}} \boldsymbol{\phi}.\nabla^2 \mathbf{z}\, d\Omega \quad \text{for all} \quad \boldsymbol{\phi} \in H_3^1(\Omega_{S'}) \tag{A 1}$$

such that $\boldsymbol{\phi}|_{\partial\Omega_{S'}} = 0, \quad \nabla.\boldsymbol{\phi} = 0.$

Let $\hat{\boldsymbol{\phi}}$ be the extension of $\boldsymbol{\phi}$ by zero in $\Omega_0 = \Omega_S \cup \mathring{\bar{S}}$. From theorem I.11.4 in Lions & Magenes (1967), $\hat{\boldsymbol{\phi}} \in H_3^1(\Omega_0)$; therefore, replacing $\boldsymbol{\phi}$ by $\tilde{\mathbf{v}}^{S'}$ (the extension by 0 of $\mathbf{v}^{S'}$) in (A 1) we obtain

$$\int_{\Omega_0} \sum_{i,j=1}^{3} \left(\frac{\partial \tilde{v}_i^{S'}}{\partial x_j}\right)^2 d\Omega = \int_{\Omega_0} \tilde{\mathbf{v}}^{S'}.\nabla^2 \mathbf{z}\, d\Omega, \tag{A 2}$$

which, from I.1.7 in Ladyzhenskaya (1963), implies that $\{\tilde{\mathbf{v}}^{S'}\}$ is bounded in $H_3^1(\Omega_0)$. As every bounded set is weakly compact, we can extract a subsequence α_i such that $\{\tilde{\mathbf{v}}^{S'}\}_i$ converges weakly to $\mathbf{w}$, say. It remains to prove that $\mathbf{w}$ is a solution of (A 1) with $S' = S$. Suppose it is not; then there exists an ϵ and an $\boldsymbol{\phi} \in H_3^1(\Omega_S)$, with $\boldsymbol{\phi}|_{\partial\Omega_S} = 0$ and $\nabla.\boldsymbol{\phi} = 0$, such that

$$\left|\int_{\Omega_S} \sum_{i,j=1}^{3} \frac{\partial w_i}{\partial x_j}\frac{\partial \phi_i}{\partial x_j}\, d\Omega - \int_{\Omega_S} \boldsymbol{\phi}.\nabla^2 \mathbf{z}\, d\Omega\right| \geqslant \epsilon > 0, \tag{A 3}$$

or equivalently,

$$\left|\int_{\Omega_0} \sum_{i,j=1}^{3} \frac{\partial \tilde{w}_i}{\partial x_j}\frac{\partial \tilde{\phi}_i}{\partial x_j}\, d\Omega - \int_{\Omega_0} \hat{\boldsymbol{\phi}}.\nabla^2 \mathbf{z}\, d\Omega\right| \geqslant \epsilon > 0, \tag{A 4}$$

and from the weak convergence of $\mathbf{v}^{S'}$ to $\mathbf{w}$,

$$\left|\int_{\Omega_0} \sum_{i,j=1}^{3} \frac{\partial \tilde{v}_i^{S'}}{\partial x_j}\frac{\partial \tilde{\phi}_i}{\partial x_j}\, d\Omega - \int_{\Omega_0} \hat{\boldsymbol{\phi}}.\nabla^2 \mathbf{z}\, d\Omega\right| \geqslant \frac{\epsilon}{2} \tag{A 5}$$

for α_i sufficiently small. Since S' converges to S, there exists an open set $\mathcal{O}$ in $\Omega_S \cap \Omega_{S'}$ such that $|\hat{\boldsymbol{\phi}}| > 0$ in $\mathcal{O}$. If χ is a smooth function into R with compact support in $\mathcal{O}$ and such that $\nabla.\hat{\boldsymbol{\phi}}\chi = 0$ then (A 1) holds with $\boldsymbol{\phi} = \hat{\boldsymbol{\phi}}\chi$ and it is contradicted by (A 5).

$\|\partial\mathbf{U}/\partial n\| = 0$ on S implies zero drag, in an unbounded domain (S axisymmetric)

The drag force F on a body with surface S is computed from

$$\mathbf{F} = \int_S (-p\mathbf{I} + \boldsymbol{\sigma})\, d\mathbf{S},$$

where $\mathbf{I}$ is the identity tensor and $\boldsymbol{\sigma}$ is as in (4.14) (stress tensor). $\nabla.\mathbf{U} = 0$ and $\|\partial\mathbf{U}/\partial n\| = 0$ implies that $\boldsymbol{\sigma}|_S = 0$. If ψ is the stream function of the problem, then $E^2\psi$ is a solution of

$$E^2(E^2\psi) = 0; \quad E^2\psi|_S = 0, \quad E^2\psi|_\infty = 0.$$

Hence $E^2\psi \equiv 0$, which implies (from 4.15.1 in Happel & Brenner 1965) that p is constant. Hence $\mathbf{F} = 0$.

$\mathbf{U}^S$ can be extended in $H_3^2(\Omega)$, slightly inside S

To extend $\mathbf{U}^S$ slightly inside S in $H_3^2(\Omega)$, we must look for a solution $\mathbf{U}$ of

$$\nabla^2\mathbf{U} = \nabla p, \quad \nabla.\mathbf{U} = 0 \quad \text{in} \quad \mathcal{O}; \quad \mathbf{U}|_S = 0, \quad \left.\frac{\partial\mathbf{U}}{\partial n_i}\right|_S = -\left.\frac{\partial\mathbf{U}^S}{\partial n_e}\right|_S, \quad \left.\frac{\partial^2\mathbf{U}}{\partial n_i^2}\right|_S = \left.\frac{\partial^2\mathbf{U}^S}{\partial n_e^2}\right|_S, \tag{A 6}$$

where $\mathcal{O}$ is an open set of R^3 with boundary $S \cup G$, where $G \subset \mathring{S}$ with G regular; $\mathbf{n}_i$ and $\mathbf{n}_e$ are the inward and outward normals of S.

From theorem I.8.2 in Lions & Magenes (1967) and theorem I.(2.2).1 in Ladyzhenskaya (1963), there exists a $\boldsymbol{\phi} \in H_3^2(\mathcal{O})$ with

$$\boldsymbol{\phi}|_S = 0, \quad \left.\frac{\partial\boldsymbol{\phi}}{\partial n_i}\right|_S = -\left.\frac{\partial\mathbf{U}^S}{\partial n_e}\right|_S, \quad \left.\frac{\partial^2\boldsymbol{\phi}}{\partial n_i^2}\right|_S = \left.\frac{\partial^2\mathbf{U}^S}{\partial n_e^2}\right|_S, \quad \nabla.\boldsymbol{\phi} = 0. \tag{A 7}$$

Therefore, if we let $\mathbf{w} = \mathbf{U} - \boldsymbol{\phi}$, we must show that

$$\nabla^2\mathbf{w} - \nabla p = \mathbf{f}, \quad \nabla.\mathbf{w} = 0 \quad \text{in} \quad \mathcal{O}; \quad \mathbf{w}|_S = 0, \quad \left.\frac{\partial\mathbf{w}}{\partial n}\right|_S = 0, \quad \left.\frac{\partial^2\mathbf{w}}{\partial n^2}\right|_S = 0 \tag{A 8}$$

has at least one solution in $H_3^2(\mathcal{O})$. This is done by a variational method which can be shown to work by a straightforward adaptation to our problem of the proofs of lemma 4.8.1. and remark 4.8.3 in Lattes & Lions (1969).

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